Discreteness of the spectrum of the Laplace-Beltrami operator

Sidi Mohammed Boughalem (shinokiz@gmail.com)

06. December 2018

In this talk, we will prove some properties about the spectrum of the unbounded symmetric operator Δ_k on the Hilbert space $L^2({}_\Gamma\backslash^{\mathcal{H}}, \chi, k)$ when ${}_\Gamma\backslash^{\mathcal{H}}$ is compact.
To do so, we will prove some facts about compact operators. Then we will introduce some *integral operators* that are self-adjoint, compact and that will commute with Δ. Finally, we will deduce the spectral theorem for Δ from the well-known spectral theorem for compact operators.

1 Preliminaries

Definition 1.1 (Linear, bounded, compact operators). *Let $\mathfrak{h}$ be a separable Hilbert space, $\mathcal{L}(\mathfrak{h})$ the vector space of linear operators $T : \mathfrak{h} \to \mathfrak{h}$.*

1. *an operator $T \in \mathcal{L}(\mathfrak{h})$ is said to be bounded if there exist a constant C such that*

$$|Tx| \leq C|x| \quad \forall x \in \mathfrak{h}$$

   The smallest such C is called the norm of the operator, and is denoted $|T|$. A bounded operator is continuous.

2. *A bounded operator $T \in \mathcal{L}(\mathfrak{h})$ is said to be self-adjoint if*

$$\langle Tf, g \rangle = \langle f, Tg \rangle \quad \forall f, g \in \mathfrak{h}$$

3. *an $f \in \mathfrak{h}$ is said to be an eigenvector of an operator T with eigenvalue λ if $f \neq 0$ and*

$$Tf = \lambda f$$

   Given λ, the set of eigenvectors with eigenvalue λ is called the λ-eigenspace and is noted

$$E_\lambda = \{f \in \mathfrak{h} \, , \, Tf = \lambda f\}$$

Seminar „Automorphic forms and Representation theory“, WS 2018/19, Universität Regensburg

Moreover, if $T \in \mathcal{L}(\mathfrak{h})$ is bounded and self-adjoint, then $\lambda \in \mathbb{R}$ and its eigenspaces are orthogonal.

4. *an operator $T \in \mathcal{L}(\mathfrak{h})$ is said to be compact if it maps bounded sets to compact sets. a compact operator is automatically bounded and continuous.*

Since $\mathfrak{h}$ is separable, we will use the sequential characterization of compactness : A linear operator $T \in \mathcal{L}(\mathfrak{h})$ is compact if and only if it is *sequentially compact* : For every sequence $(x_n) \subset \mathfrak{h}$ of unit vectors, there is a subsequence $(x_{n_k}) \subset \mathfrak{h}$ such that $T(x_{n_k})$ is convergent. We will use this characterization of compactness to prove the following :

Theorem 1.2 (Spectral theorem for compact operators)**.** *Let $T \in \mathcal{L}(\mathfrak{h})$ be a compact self-adjoint operator, then $\mathfrak{h}$ has an orthonormal basis $\{f_i\}_{i\geq 1}$ consisting of eigenvectors of T, so that*

$$Tf_i = \lambda_i f_i \quad , \quad \lambda_i \xrightarrow[i\to\infty]{} 0$$

In particular, the eigenspaces E_{λ_i} are finite dimensional.

Proof. Let $T \in \mathcal{L}(\mathfrak{h})$ be a self-adjoint, compact operator, we first show that

$$|T| = \sup_{\substack{x\in\mathfrak{h}\\ x\neq 0}} \frac{\langle Tx, x\rangle}{\langle x, x\rangle}$$

Let $0 \neq x \in \mathfrak{h}$, put $B := \frac{|\langle Tx,x\rangle|}{\langle x,x\rangle}$.
On one hand, one has

$$|\langle Tx, x\rangle| \leq |Tx||x| \leq |T||x|^2 = |T|\langle x, x\rangle \Rightarrow B \leq |T|$$

On the other hand, let $\lambda > 0$, as $\langle Tx, Tx\rangle = \langle T^2x, x\rangle$, one has

$$\begin{aligned}
\langle Tx, Tx\rangle &= \frac{1}{4}|2\langle Tx, Tx\rangle + 2\langle T^2x, x\rangle| \\
&\leq \frac{1}{4}|\langle T(\lambda x + \lambda^{-1}Tx), \lambda x + \lambda^{-1}Tx\rangle - \langle T(\lambda x - \lambda^{-1}Tx), \lambda x - \lambda^{-1}Tx\rangle| \\
&\leq \frac{1}{4}|\langle T(\lambda x + \lambda^{-1}Tx), \lambda x + \lambda^{-1}Tx\rangle| + \frac{1}{4}|\langle T(\lambda x - \lambda^{-1}Tx), \lambda x - \lambda^{-1}Tx\rangle| \\
&\leq \frac{B}{4}\langle \lambda x + \lambda^{-1}Tx, \lambda x + \lambda^{-1}Tx\rangle + \frac{B}{4}\langle \lambda x - \lambda^{-1}Tx, \lambda x - \lambda^{-1}Tx\rangle \\
&= \frac{B}{2}\lambda^2\langle x, x\rangle + \frac{B}{2}\langle \lambda Tx, Tx\rangle.
\end{aligned}$$

Put $\lambda = (\frac{|Tx|}{|x|})^{\frac{1}{2}}$, we get

$$|Tx|^2 \leq B|x||Tx| \Rightarrow |Tx| \leq B|x| \Rightarrow |T| \leq B$$

Now let $\Sigma = \{E_{\lambda_i} \subset \mathfrak{h}, i = 1, 2...\}$. By ordering Σ by inclusion, Zorn's Lemma implies that it has a maximal element, say E. We show that

$$\mathfrak{h} = \bigoplus_{i\geq 1} E_{\lambda_i}$$

– the $(E_{\lambda_i})_{i\leq 1}$ are mutually orthogonal : Let $u \in E_{\lambda_i}$, $v \in E_{\lambda_j}$, $i \neq j$

$$\begin{aligned} \lambda_i\langle u, v\rangle = \langle Tu, v\rangle = \langle u, Tv\rangle = \lambda_j\langle u, v\rangle \\ \Rightarrow \langle u, v\rangle = 0 \end{aligned}$$

– $Span_{\mathfrak{h}}(E)$ is dense in $\mathfrak{h}$:

let $V = \overline{Span_{\mathfrak{h}}(E)}$ and suppose $0 \subsetneq V \subsetneq \mathfrak{h}$, there exist then a $\mathfrak{h}_0 = V^\perp$ such that

$$\mathfrak{h} = V \oplus \mathfrak{h}_0, \quad \text{and} \quad \mathfrak{h}_0 \neq 0$$

As $T(V^\perp) \subset V^\perp$ and $T_{/\mathfrak{h}_0}$ is compact self-adjoint, we only have to show that T has an eigenvector in $\mathfrak{h}_0$, this will contradict the maximality of Σ. Since T is compact, one has a sequence $(x_i)_{i\geq 1}$ of unit vectors such that

$$|\langle Tx_i, x_i\rangle| \xrightarrow[i\to\infty]{} |T|$$

As $\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$, the $\langle Tx_i, x_i\rangle$ are real. We may thus replace our sequence by a sub-sequence $\langle Tx_i, x_i\rangle \xrightarrow[i\to\infty]{} \lambda = \pm|T|$. If $\lambda = 0$ then we are done, since $T = 0$ has eigenvectors. If not, as T is compact, we have a subsequence (x_i) such that $Tx_i \xrightarrow[i\to\infty]{} v$ First, we have

$$|\langle Tx_i, x_i\rangle| \leq |Tx_i||x_i| = |Tx_i| \leq |T||x_i| = |\lambda|$$

Now, one sees that

$$\begin{aligned} |\lambda x_i - Tx_i|^2 = \langle \lambda x_i - Tx_i, \lambda x_i - Tx_i\rangle = \lambda^2|x_i| + |Tx_i|^2 - 2\langle Tx_i, x_i\rangle \\ \xrightarrow[i\to\infty]{} \lambda^2 + \lambda^2 - 2\lambda^2 = 0. \end{aligned}$$

Hence

$$\lambda x_i \xrightarrow[i\to\infty]{} Tx_i \xrightarrow[i\to\infty]{} v \Rightarrow x_i \xrightarrow[i\to\infty]{} \lambda^{-1}v$$

But, since T is continuous, one has $Tx_i \xrightarrow[i\to\infty]{} T(\lambda^{-1}v) = \lambda^{-1}Tv$
thus one gets finally that

$$\lambda v = Tv$$

and hence, the contradiction.

Now let $\{f_i\}_{i\geq 1}$ be an ONB of $\mathfrak{h}$, and $(\lambda_i)_{i\geq 1}$ the associated eigenvalues. We prove that

$$\lambda_i \xrightarrow[i\to\infty]{} 0$$

Suppose it is not the case, then $\exists\epsilon > 0$, $(f'_i)_{i\geq 1}$ a sub sequence of $(f'_i)_{i\geq 1}$ such that $Tf'_i > \epsilon$. Since all the Tf_i are orthogonal, Tf_i can not have any convergent subsequence which contradicts the compactness of T. $\square$

Exercise 1.3. *Let* $K(\mathfrak{h}) = \{T \in \mathcal{L}(\mathfrak{h}) \; / \; T \text{ is compact } \}$.
Show that $K(\mathfrak{h})$ *is a closed linear subspace of* $\mathcal{L}(\mathfrak{h})$.
In particular, for every convergent sequence T_n *in* $K(\mathfrak{h})$ *one has*

$$T_n \xrightarrow[n\to\infty]{} T \in K(\mathfrak{h})$$

Theorem 1.4 (Hilbert-Schmidt operators)**.** *Let* X *be a locally compact Borel-measurable space,* $\mathfrak{h} = L^2(X)$. *Let* $K \in L^2(X \times X)$. *Then the operator*

$$(Tf)(x) = \int\limits_X K(x,y)f(y)dy$$

is a well defined, compact operator on $\mathfrak{h}$.

Proof. If $\{f_i\}_{i\geq 1}$ is an ONB of $L^2(X)$ then one has an ONB for $L^2(X \times X)$, namely $\{f_i, f_j\}_{i,j\geq 1}$.

Write

$$k(x,y) = \sum_{i,j\geq 1} k_{i,j} f_i(x) f_j(y), \quad k_{i,j} = \iint\limits_X k(x,y) f_i(x) f_j(y)\, dx\, dy$$

Define

$$k_n(x,y) = \sum_{i=1}^{n} \sum_{j\geq 1} k_{i,j} f_i(x) f_j(y)$$

and the following sequence of operators

$$T_n f = \int\limits_X k_n(x,y) f(y)\, dy$$

The T_n are compact since they map $L^2(X)$ to a finite dimensional subspace (spanned by the f_i for $i = 1, ..., n$) of $L^2(X)$ and thus their range is finite dimensional (every bounded linear operator of finite rank is compact). One has

$$\begin{aligned}
|(T_n - T)f|^2 &= \int\limits_X (\int\limits_X (k_n(x,y) - k(x,y)) f(y)\, dy)^2\, dx \\
&\leq \int\limits_X (\int\limits_X (k_n(x,y) - k(x,y))^2\, dy)(\int_X f(y)^2\, dy))\, dx \quad \text{(Hölder inequality)} \\
&= \int_X f(y)^2\, dy \iint\limits_X (k_n(x,y) - k(x,y))^2\, dy\, dx \\
&\leq |f|^2 (\sum_{i\geq n+1} \sum_{j\geq 1} |k_{i,j}|^2).
\end{aligned}$$

Since $|k_{i,j}|_{L^2(X\times X)} < \infty$, the above sum has to go to zero as $n \longrightarrow \infty$, hence we get that

$$|T_n - T| \xrightarrow[n\to\infty]{} 0 \Rightarrow T_n \xrightarrow[n\to\infty]{} T \in K(L^2(X)) \quad \text{(by Exercise 1.3)}$$

$\square$

Such operators are called *Hilbert-Schmidt* operators, with kernel k. In particular, if $k(x,y) = \overline{k(y,x)}$ as we will see in the next section, the spectral theorem applies and we get a decomposition of finite dimensional eigenspaces for T.

2 Spectral theory for integral operators

Now let $G = GL_2(\mathbb{R})^+$, $\Gamma \subset SL_2(\mathbb{R}) := G_1$ a discrete subgroup of G such that ${}_\Gamma\backslash^{\mathcal{H}}$ is compact. Let χ be a unitary character of Γ and $\mathfrak{h} = L^2({}_\Gamma\backslash^G, \chi)$. Let $\phi \in C_c^\infty(G)$ and let (π, H) be an arbitrary representation on a Hilbert space H. We define $\pi(\phi) \in End(H)$ by

$$\pi(\phi)f = \int_G \phi(g)\pi(g)f\,dg \quad \text{for } f \in H, g \in G$$

It is well defined since $g \longmapsto \phi(g)\pi(g)f$ is continuous in the compact support of ϕ, and thus Borel-integrable.
Moreover, one has

$$\begin{aligned}
|\pi(\phi)f| &\leq \int_G |\phi(g)\pi(g)f|\,dg \\
&\leq |f| \sup_{g\in\overline{Supp(\phi)}} |\pi(g)| \int_G |\phi(g)|\,dg \\
&\leq |f| B \int_G |\phi(g)|\,dg < \infty.
\end{aligned}$$

Thus $\pi(\phi)$ is a bounded operator.

Now let $\pi = \rho$ the regular right representation of G on $\mathfrak{h}$ (which is unitary) and define for every $\phi \in C_c^\infty(G)$ the operator $\rho(\phi) \in \mathcal{L}(\mathfrak{h})$ by

$$(\rho(\phi)f)(g) = \int_G \phi(h)(\rho(h)f)(g)dh = \int_G \phi(h)f(gh)dh$$

For this operator to be well defined, we need to show that $\rho(\phi)f$ is square integrable. We make use of the unitarity of ρ. Indeed, one has

$$\begin{aligned}
|\rho(\phi)f|^2 = \langle\, \rho(\phi)f, \rho(\phi)f\,\rangle &= \langle \int_G \phi(g)\pi(g)f\,dg, \int_G \phi(h)\pi(h)f\,dh\,\rangle \\
&= \int_G\int_G \phi(g)\overline{\phi(h)}\langle\, \rho(g)f, \rho(h)f\,\rangle\,dg\,dh.
\end{aligned}$$

Since ρ is unitary, $\rho(f), \rho(g)$ are unitary operators and therefore, by Cauchy-Schwartz

$$|\langle\, \rho(g)f, \rho(h)f\,\rangle|^2 \leq \langle\, \rho(g)f, \rho(g)f\,\rangle\langle\, \rho(h)f, \rho(h)f\,\rangle \leq \langle\, f, f\,\rangle^2$$

Thus

$$|\langle\, \rho(g)f, \rho(h)f\,\rangle| \leq \langle\, f, f\,\rangle$$

and finally
$$|\rho(\phi)f|^2 \leq |f|^2 (\int_G |\phi(g)|\, dg)^2 < \infty$$
Recall that every element of G has a representation
$$\gamma h u \in {}_{\Gamma}\backslash^{G}/_{Z^+} \quad \text{and } f(\gamma h u) = \chi(\gamma) f(h) \;\; \forall f \in C^\infty({}_\Gamma\backslash^G, \chi)$$
where $Z^+ = \langle u I_2 \rangle_{u>0}$ is the subgroup (in the center of G) of positive scalar matrices. For $\theta \in \mathbb{R}$, let $\kappa_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \in K = SO_2$

Proposition 2.1. *Let $\phi \in C_c^\infty(G)$*

(i) The operator $\rho(\phi) \in \mathcal{L}(\mathfrak{h})$ is a Hilbert-Schmidt operator. In particular, it is compact and maps $L^2({}_\Gamma\backslash^G, \chi)$ to $C^\infty({}_\Gamma\backslash^G, \chi)$.

(ii) if $\phi(g) = \overline{\phi(g^{-1})}$ then $\rho(\phi)$ is self-adjoint.

(iii) if $\phi(\kappa_\theta g) = e^{-ik\theta}\phi(g)$ then $\rho(\phi)$ maps $L^2({}_\Gamma\backslash^G, \chi)$ to $C^\infty({}_\Gamma\backslash^G, \chi, k)$.

Proof. (i) Consider
$$(\rho(\phi)f)(g) = \int_G f(gh)\phi(h)\, dh$$

First we make the substitution $h \mapsto g^{-1}h$ and use our representation of $g \in G$ as $\gamma h u \in {}_\Gamma\backslash^G/_{Z^+}$
$$\begin{aligned}(\rho(\phi)f)(g) &= \int_G f(h)\phi(g^{-1}h)\, dh \\ &= \int_{{}_\Gamma\backslash^G/_{Z^+}} \int_{Z^+} f(\gamma h u)\phi(g^{-1}\gamma h u)\, du\, dh\, d\gamma.\end{aligned}$$

This is possible since the groups involved in the decomposition are unimodular. it is mainly due to the G-invariance of the Haar measures on these cosets. As Γ is discrete, we get
$$(\rho(\phi)f)(g) = \int_{{}_\Gamma\backslash^G/_{Z^+}} \int_{Z^+} \sum_{\gamma\in\Gamma} \chi(\gamma) f(h)\phi(g^{-1}\gamma h u)\, du\, dh.$$

Let $\mathcal{F}$ be the closure of a fundamental domain for ${}_\Gamma\backslash^G/_{Z^+}$ we can write
$$\begin{aligned}(\rho(\phi)f)(g) &= \int_{\mathcal{F}} f(h) \int_{Z^+} \sum_{\gamma\in\Gamma} \chi(\gamma)\phi(g^{-1}\gamma h u)\, du\, dh \\ &= \int_{\mathcal{F}} f(h) K(g,h)\, dh.\end{aligned}$$

We are left to prove that

$$K(g,h) = \int_{Z^+} \sum_{\gamma \in \Gamma} \chi(\gamma)\phi(g^{-1}\gamma h u)\, du$$

is a *Hilbert-Schmidt* kernel, i.e. $K \in L^2(\mathcal{F} \times \mathcal{F})$. ϕ is smooth and supported in K' (compact) and since Γ is discrete, one has

$$\phi(g^{-1}\gamma h) \neq 0 \Leftrightarrow g^{-1}\gamma h \in K \Leftrightarrow \gamma \in g.K.h^{-1} \cap \Gamma \text{ which is finite}$$

Thus, we have a well behaved finite sum of smooth compactly supported functions, one can indeed deduce that $K_g(h)$ and $K_h(g)$ are smooth, and thus so is $K(g,h)$. In particular, it is square integrable on the compact $\mathcal{F} \times \mathcal{F}$.

Now let $f \in L^2({}_{\Gamma}\backslash^{G}, \chi)$, then by Cauchy-Schwartz

$$\int_{{}_{\Gamma}\backslash^{G}/_{Z^+}} |f(g)|\, dg \leq |f||1| < \infty$$

Hence, as $K(g,h)$ is smooth, the integral $\int_{\mathcal{F}} f(h)K(g,h)\, dh$ converges for every g and thus $\rho(\phi)f \in C^\infty({}_{\Gamma}\backslash^{G}, \chi)$.

(ii) Now suppose that $\phi(g) = \overline{\phi(g^{-1})}$ one sees that

$$\begin{aligned}
\overline{K(h,g)} &= \int_{Z^+} \sum_{\gamma \in \Gamma} \overline{\chi(\gamma)\phi(h^{-1}\gamma g u)}\, du \\
&= \int_{Z^+} \sum_{\gamma \in \Gamma} \overline{\chi(\gamma)}\phi(u^{-1}g^{-1}\gamma^{-1}h)\, du \\
&= \int_{Z^+} \sum_{\gamma \in \Gamma} \chi(\gamma^{-1})\phi(g^{-1}\gamma^{-1}hu^{-1})\, du \quad (\chi \text{ is unitary}) \\
&= \int_{Z^+} \sum_{\gamma \in \Gamma} \chi(\gamma)\phi(g^{-1}\gamma h u^{-1})\, du.
\end{aligned}$$

We make the change of variable $u \mapsto u^{-1}$. As Z^+ is unimodular (Abelian), its Haar measure is invariant under this change and one has

$$\overline{K(h,g)} = \int_{Z^+} \sum_{\gamma \in \Gamma} \chi(\gamma)\phi(g^{-1}\gamma h u)\, du = K(g,h).$$

Finally,

$$\begin{aligned}
\langle\, \rho(\phi)f), f'\,\rangle &= \int\limits_{\mathcal{F}} (\int\limits_{\mathcal{F}} f(h)K(g,h)\,dh) f'(g)\,dg \\
&= \iint\limits_{\mathcal{F}} f(h)\overline{f'(g)K(h,g)}\,dh\,dg \\
&= \int\limits_{\mathcal{F}} f(h)\overline{(\int\limits_{\mathcal{F}} f'(g)K(h,g)\,dg)}\,dh \\
&= \langle\, f, \rho(\phi)f'\,\rangle.
\end{aligned}$$

(iii) Suppose $\phi(\kappa_\theta g) = e^{-ik\theta}\phi(g)$, than one has

$$(\rho(\phi)f)(g\kappa_\theta) = \int\limits_G f((g\kappa_\theta h)\phi(h)\,dh$$

By making the change of variable $h \mapsto \kappa_\theta^{-1}h$, one has

$$\begin{aligned}
(\rho(\phi)f)(g) &= \int\limits_G f((gh)\phi(\kappa_\theta^{-1}h)\,dh \\
&= e^{ik\theta}\int\limits_G f(gh)\phi(h)\,dh = e^{ik\theta}(\rho(\phi)f)(g).
\end{aligned}$$

Thus $\rho(\phi)f \in C^\infty({}_\Gamma\backslash^G, \chi, k)$.

$\square$

Recall that $C^\infty({}_\Gamma\backslash^G, \chi, k) = \{f \in C^\infty({}_\Gamma\backslash^G, \chi)/f(g\kappa_\theta) = \rho(\kappa_\theta)f = e^{-ik\theta}\phi(g)\}$. We have seen before that with a convenient choice of ϕ, we can restrict the range of our operator $\rho(\phi)$ enough to $C^\infty({}_\Gamma\backslash^G, \chi, k)$ so that it commutes with Δ. We shall see that with a right adjustment we might be able to choose such ϕ.

Let $\pi : G \to End(H)$ be a unitary representation of G on a Hilbert space H. By the same reasoning as in the proof above, one sees that

$$
\begin{aligned}
\langle\, \pi(\phi)v, w \,\rangle &= \int_{\mathcal{G}} \phi(g)\langle\, \pi(g)v, w \,\rangle \, dg \\
&= \int_{\mathcal{G}} \phi(g)\langle\, v, \pi(g^{-1})w \,\rangle \, dg \quad (\pi(g) \text{ is unitary}) \\
&= \langle\, v, \int_{\mathcal{G}} \overline{\phi(g)}\pi(g^{-1})w \, dg \,\rangle \\
&= \langle\, v, \int_{\mathcal{G}} \phi(g^{-1})\pi(g^{-1})w \, dg \,\rangle \qquad (\overline{\phi(g^{-1})} = \phi(g)) \\
&= \langle\, v, \pi(\phi)w \,\rangle \qquad (\text{ by making } g \mapsto g^{-1})
\end{aligned}
$$

Lemma 2.2. *Let $\pi : G \to End(H)$ be a unitary representation of G on a Hilbert space H and let $0 \neq f \in H$.*

(i) Let $\epsilon > 0$, then there exists $\phi \in C_c^\infty(G)$ such that $\pi(\phi)$ is self-adjoint and $|\pi(\phi)f - f| < \epsilon$. In particular, if $\epsilon < |f|$ then

$$\pi(\phi)f \neq 0$$

(ii) If $\pi(\kappa_\theta)f = e^{ik\theta}f$ for all $\kappa_\theta \in K = SO_2$ then we may choose ϕ so that

$$\phi(\kappa_\theta g) = \phi(g\kappa_\theta) = e^{ik\theta}\phi(g)$$

Proof. (i) As $g \mapsto \pi(\phi)f$ is continuous, we can find an open neighbourhood $U \in \mathcal{V}_{I_2}$ such that

$$|\pi(g)f - f| < \epsilon \;\; \forall g \in U$$

Let $\phi_0 \in C_c^\infty(\mathbb{R})$, ϕ_0 positive and $Supp(\phi_0) \subset U$ such that $\int_G \phi_0(g)\,dg = 1$. We assume that $\phi_0(g) = \phi_0(g^{-1})$, by Proposition 2.1 (ii), $\pi(\phi_0)$ is self-adjoint. We have

$$|\pi(\phi_0)f - f| = |\int_G \phi_0(g)(\pi(g)f - f)\,dg| \leq \int_G \phi_0(g)|\pi(g)f - f|\,dg < \epsilon$$

Moreover, if $\epsilon < |f|$ then clearly $\pi(\phi_0)f \neq 0$.

(ii) Now we shall construct carefully our ϕ_0. Observe first that the map

$$
\begin{aligned}
\sigma : G \times K &\longrightarrow G \\
(g, \kappa) &\longmapsto \kappa g \kappa^{-1}
\end{aligned}
$$

is continuous, hence σ^{-1} is open in $G \times K$. As $(1, \kappa) \in \sigma^{-1}(U)$ for all $\kappa \in K$, there exists an open neighbourhood $U' \in \mathcal{V}_{(I_2,\kappa)}$ such that $U' = V_\kappa \times W_\kappa \subset G \times K$.

As K is compact and W_κ is an open cover of K, there exists an $r>0$ such that $\{W_{\kappa_i}\}_{0\leq i\leq r}$ is a finite open cover of K. By taking

$$V=\bigcap_{i=1}^{r} V_{\kappa_i} \Rightarrow \kappa V\kappa^{-1}\subset U \quad \forall \kappa\in K$$

Thus we obtain a neighbourhood in the identity of G with the desired property. Now let $\phi_1\in C_c^\infty(\mathbb{R})$, ϕ_1 positive and $Supp(\phi_1)\subset V$ such that $\phi_1(g)=\phi_1(g^{-1})$ and let

$$\phi_0(g)=\int_K \phi_1(\kappa g\kappa^{-1})\,dg$$

Then ϕ_0 is a positive function with support in U that satisfies $\phi_0(g)=\phi_0(g^{-1})$ and $\phi_0(\kappa g\kappa^{-1})=\phi_0(g)$ for all $\kappa\in K$. Assume now that $\pi(\kappa_\theta)f=e^{ik\theta}f$, since $G=GK$ and K is compact, one has

$$\begin{aligned}
\pi(\phi_0)f &= \int_G \phi_0(h)\pi(h)f\,dh=\int_G\int_K \phi_0(h\kappa)\pi(h\kappa)f\,dh\,d\kappa\\
&=\int_G \frac{1}{2\pi}\int_0^{2\pi}\phi_0(h\kappa_\theta)\pi(h\kappa_\theta)f\,dh\,d\theta=\int_G\frac{1}{2\pi}\int_0^{2\pi}\phi_0(h\kappa_\theta)\pi(h)\pi(\kappa_\theta)f\,dh\,d\theta\\
&=\int_G [\frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta}\phi_0(h\kappa_\theta)\,d\theta\,]\pi(h)f\,dh=\pi(\phi)f.
\end{aligned}$$

where

$$\phi(g)=\frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta}\phi_0(h\kappa_\theta)\,d\theta$$

and clearly

$$\phi(\kappa_\theta g)=\phi(g\kappa_\theta)=e^{ik\theta}\phi(g)\quad \text{and } \phi(g)=\overline{\phi(g^{-1})}$$

□

A representation (π,H) of a group G on a Hilbert space H is said to be *irreducible* if H has no proper non-zero closed subspace that is invariant under π.
If π is unitary, V is a proper non-zero π-invariant subspace of H, then there exists a proper non-zero closed subspace $V^\perp$ such that

$$H=V\oplus V^\perp$$

We start with the following :

Exercise 2.3. *Let H be a Hilbert space and $\pi: H\to End(H)$ be a unitary representation. Let*

$$H_k=\{f\in H/\pi(\kappa_\theta)f=e^{2i\pi k\theta}f\}$$

Show that the H_k are orthogonal, and that

$$H = \bigoplus_{k\in\mathbb{Z}} H_k$$

We will use this result in the next proposition

Proposition 2.4. *Let H be a non-zero closed subspace of $\mathfrak{h}$, which is closed under the action of G. Then we have a Hilbert space decomposition*

$$H = \bigoplus_{k\in\mathbb{Z}} H_k$$

where $H_k = \{f \in H/\rho(\kappa_\theta)f = e^{2i\pi k\theta} f\}$.

Moreover, if $H_k \neq 0$ then Δ has a non-zero eigenvector in $H_k \cap C^\infty({}_\Gamma\backslash^G, \chi)$.

Proof. We know that for $g \in G$, $\Delta \circ \rho(g) = \rho(g) \circ \Delta$. Thus, one has

$$\begin{aligned}(\Delta \circ \rho(\phi))f &= \int_G \phi(g)(\Delta \circ \rho(g))f\,dg \\ &= \int_G \phi(g)(\rho(g) \circ \Delta)f\,dg = (\rho(\phi) \circ \Delta)f.\end{aligned}$$

Now let H be a closed G-invariant subspace of $\mathfrak{h}$, from Exercise (2.3), one has a Hilbert space decomposition $H = \bigoplus H_k$ where $\rho(\kappa_\theta)f = e^{2i\pi k\theta} f$ for $f \in H_k$. We choose a k such that $H_k \neq 0$ and let $0 \neq f_0 \in H_k$.
From Lemma (2.2) there exists a $\phi \in C_c^\infty(G)$ such that $\rho(\phi)f_0 \neq 0$ and $\phi(\kappa_\theta g) = \phi(g\kappa_\theta) = e^{ik\theta}\phi(g)$. By proposition (2.1) $\rho(\phi)$ is a self-adjoint compact operator that maps H into $H \cap C^\infty({}_\Gamma\backslash^G, \chi, k) = H_k \cap C^\infty({}_\Gamma\backslash^G, \chi)$. By the spectral theorem, H has an ONB $\{f_i\}_{i\geq 1}$ with respective eigenvalues λ_i. Pick a $0 \neq \lambda \in H_k$ then E_{λ_i} is finite dimensional and since Δ commutes with $\rho(\phi)$, E_λ is invariant under Δ.
By the fundamental theorem of Algebra, every linear operator on a finite dimensional complex vector space has an eigenvalue, thus $\rho(\phi)$ has at least an eigenvector in $H_k \cap C^\infty({}_\Gamma\backslash^G, \chi)$. ◻

Theorem 2.5. *The space $\mathfrak{h} = L^2({}_\Gamma\backslash^G, \chi)$ decomposes into a Hilbert space direct sum of subspaces that are invariant and irreducible under the right regular representation ρ.*

Proof. Let Σ be the set of all sets S of irreducible invariant subspaces of $L^2({}_\Gamma\backslash^G, \chi)$ such that the elements of S_i are mutually orthogonal. By ordering Σ by inclusion, Zorn's lemma implies that Σ has a maximal element, say S. Let H' be the orthogonal complement of the closure of the direct sum of the elements of S. We want to show that $H' = 0$.
Suppose not, and let $0 \neq f \in H'$. The goal is to construct an irreducible subspace of

H' which will contradict the maximality of S.
Let $\phi \in C_c^\infty(G)$ such that $\rho(\phi)$ is self-adjoint, and $\rho(\phi)f \neq 0$ (Lemma (2.2)). By the spectral theorem, one has a non-zero eigenvalue λ with a finite dimensional eigenspace $E_\lambda \subset H'$.
Let Ł$_i$ be a non-zero invariant subspace of E_λ under $\rho(\phi)$ and consider
$L_0 = \min_i \{E_\lambda \cap L_i\}$ (such a subspace exist since the L_i are finite dimensional)
Let

$$V = \bigcap \{W \subset H' \ / \ L_0 = E_\lambda \cap W\} \subset H'$$

We only need to show that V is irreducible, which will as expected contradict the maximality of S.
Suppose it is not, and consider V_1, V_2 invariant subspaces of V such that $V = V_1 \oplus V_2$.
Let $0 \neq f_0 \in V$ such that $f_0 = f_1 + f_2$ with $f_i \in V_i$ for $i = 1, 2$. By definition of ρ, every closed G-invariant subspace V_i is also invariant under $\rho(\phi)$.
One gets

$$(\rho(\phi)f_1 - \lambda f_1) + (\rho(\phi)f_2 - \lambda f_2) = \rho(\phi)f_0 - \lambda f_0 = \rho(\phi)f_0 - \rho(\phi)f_0 = 0$$

Thus $\rho(\phi)f_i = \lambda f_i$ for $i = 1, 2$.
Without loss of generality, suppose $f_1 \neq 0$. Then $f_1 \in E_\lambda \cap V_1 \subset L_0$, and by minimality, $E_\lambda \cap V_1 = L_0$.
But since $V \subset V_1$ that would imply that $V = V_1$ which is impossible since V_1 is proper.
Hence V is irreducible and thus the result. □

3 Spectral decomposition

Let $k \in \mathbb{R}$ and $\sigma : \kappa_\theta \mapsto e^{ik\theta}$ be the character of $K = SO_2$. Let $R_\sigma := C^\infty({}_K\backslash^G/_K, \sigma)$ be the **commutative** convolution subring of smooth compactly supported functions ϕ that satisfy

$$\phi(\kappa_1 g \kappa_2) = \sigma(\kappa_1)\phi(g)\sigma(\kappa_2) \quad \forall \kappa_1, \kappa_2 \in K, \ \ \forall g \in G$$

We define a *character* of this ring to be a ring homomorphism into $\mathbb{C}$.

Theorem 3.1. *Let $\xi \in \mathrm{Hom}_R(R_\sigma, \mathbb{C})$ be a character of R_σ, and let H_ξ be the space of $f \in L^2({}_\Gamma\backslash^G, \chi, k)$ that satisfies $\rho(\phi)f = \xi(\phi)f$ for all $\phi \in R_\sigma$.*

(i) The space H_ξ is a finite dimensional subspaces of $C^\infty({}_\Gamma\backslash^G, \chi, k)$

(ii) For all $\xi \neq \eta$ characters of R_σ, $H_\xi \perp H_\eta$.

(iii)

$$L^2({}_\Gamma\backslash^G, \chi, k) = \bigoplus_{\substack{\xi \in \mathrm{Hom}_R(R_\sigma, \mathbb{C}) \\ H_\xi \neq 0}} H_\xi$$

Proof. (i) Suppose $0 \neq f \in H_\xi$. By Lemma (2.2), there exists $\phi \in R_\sigma$ such that

$$\rho(\phi)f = \xi(\phi)f \neq 0 \quad \Rightarrow \quad \xi(\phi) \neq 0$$

Put $\lambda = \xi(\phi) \in \mathbb{C}^*$. By the spectral theorem, the eigenspace E_λ is finite dimensional, and since by definition $H_\xi \subset E_\lambda$, we have the result.

(ii) Now let $\eta, \xi \in \mathrm{Hom}_R(R_\sigma, \mathbb{C})$ such that $\eta \neq \xi$. Then there exists $\phi \in R_\sigma$ such that $\eta(\phi) \neq \xi(\phi)$.
We write $\phi = \phi_1 + i\phi_2$, where

$$\phi_1 = \frac{1}{2}(\phi(g) + \overline{\phi(g^{-1})}), \quad \phi_2 = \frac{1}{2i}(\phi(g) - \overline{\phi(g^{-1})})$$

Since

$$\overline{\phi_1(g^{-1})} = \frac{1}{2}(\overline{\phi(g^{-1})} + \phi(g)) = \phi_1(g)$$
$$\overline{\phi_2(g^{-1})} = -\frac{1}{2i}(\overline{\phi(g^{-1})} - \phi(g)) = \frac{1}{2i}(\phi(g) - \overline{\phi(g^{-1})} = \phi_2(g)$$

By proposition (2.1) (ii), $\rho(\phi_1), \rho(\phi_2)$ are self-adjoint.
Now without loss of generality, assume $\rho(\phi)$ is self-adjoint (by simply choosing either ϕ_1 or ϕ_2) then

$$\begin{array}{ccc} E_{\xi(\phi)} & \perp & E_{\eta(\phi)} \\ \cup & & \cup \\ H_\xi & \perp & H_\eta \end{array}$$

(iii) As done before, we suppose that there exists an orthogonal supplement H' such that
$$L^2(\Gamma\backslash^G, \chi, k) = H' \oplus \bigoplus_{\substack{\xi\in \mathrm{Hom}_R(R_\sigma,\mathbb{C})\\ H(\xi)\neq 0}} H_\xi$$
And we show that $H' = 0$.
Suppose it's not, and let $0 \neq f \in H'$. By lemma (2.2) $\exists \phi_0 \in R_\sigma$ such that $|\rho(\phi)f - f| < \epsilon$. Hence we can arrange so that $\rho(\phi)f$ and f are not orthogonal. By the spectral theorem, we have an ONB $\{f_i\}_{i\geq 1}$ of eigenvectors and thus the spectral expansion
$$\rho(\phi)f = \sum_{i\geq 1} \lambda_i f_i$$
As $\rho(\phi)f$ is not orthogonal to f, there exists an $i \geq 1$ such that f_i is not orthogonal to f.
Let $V = E_{\lambda_i}$ be the eigenspace consisting of such eigenvector. Then V is finite dimensional and since R_σ is commutative, V is invariant under $\rho(\phi)$ for all $\phi \in R_\sigma$. Indeed, one has
$$\begin{aligned}\pi(\phi)(\lambda_i f_i) &= \pi(\phi)(\pi(\phi_0)f_i) = \pi(\phi\phi_0)f_i)\\ &= \pi(\phi_0\phi)f_i) = \pi(\phi_0)(\pi(\phi)f_i) \in E_{\lambda_i}\end{aligned}$$
Thus, we have
$$V = \bigoplus_{\substack{\xi\in \mathrm{Hom}_R(R_\sigma,\mathbb{C})\\ \xi(\phi_0)=\lambda_i}} H_\xi$$
But since f is not orthogonal to f_i, it can not be orthogonal to all these spaces, thus $H' = 0$ and hence our result.

□

We are finally able to present a version of the spectral theorem for Δ_k, mainly we have the following result

Corollary 3.2 (Main result)**.** *The space $L^2(\Gamma\backslash^{\mathcal{H}}, \chi, k)$ decomposes into a Hilbert space direct sum of eigenspaces for Δ_k.*

Recall that
$$\begin{array}{ccc} L^2(\Gamma\backslash^{\mathcal{H}}, \chi, k) & \cong & L^2(\Gamma\backslash^G, \chi, k) \\ \Delta_k & \longleftrightarrow & \Delta \end{array}$$

To see how this is a direct consequence of Theorem (3.1), observe that since Δ commutes with the operators $\rho(\phi)$ on R_σ, the spaces H_ξ are Δ-invariant and Δ induces a self-adjoint compact operator on each of them, and thus each of the H_ξ decomposes into a direct sum of Δ-Eigenspaces.

References

[1] D. Bump. *Automorphic Forms and Representations*, Cambridge Studies in Advanced Mathematics 55, 1998.